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ATOMIC SURFACES, TILINGS AND COINCIDENCE I IRREDUCIBLE CASE

BY

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ABSTRACT

An irreducible Pisot substitution defines a graph-directed iterated function system. The invariant sets of this iterated function system are called the atomic surfaces. In this paper, a new tiling of atomic surfaces, which contains Thurston's β-tiling as a subclass, is constructed. Related tiling and dynamical properties are studied. Based on the coincidence condition defined by Dekking [Dek], we introduce the **super-coincidence condition**. It is shown that the super-coincidence condition governs the tiling and dynamical properties of atomic surfaces. We conjecture that every Pisot substitution satisfies the super-coincidence condition.

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1. Introduction

1.1. A BRIEF REVIEW. A substitution is a morphism of the free monoid on a finite alphabet of size d. In a specific case (the so-called irreducible and unimodular Pisot case), it can be associated with a **graph-directed Iterated Function System** (graph-IFS). Let $\{X_i\}_{1\leq i\leq d}$ be the invariant sets of this graph-IFS; then the set $X = \bigcup_{i=1}^{d} X_i$ is called the **atomic surface** of the substitution and $\{X_i\}_{1\leq i\leq d}$ are called the **partial atomic surfaces**. (See Figure 1.)

The atomic surfaces of substitutions were studied first by Rauzy ([Rau]), investigating a particular but essential substitution

$$1 \mapsto 12, \ 2 \mapsto 13, \ 3 \mapsto 1$$

which is called the Rauzy substitution. Rauzy showed that the atomic surface of Rauzy substitution is related to fractal geometry, number theory, tiling theory, ergodic theory and dynamical systems. It illustrates the connections between these different areas. Since Rauzy, atomic surfaces, also called **Rauzy fractals**, have been studied by many authors ([Luc, AI, SW, CS, Sie1, Ak1, Ak2, BD, HZ, Hos2, IK, Mes, Pra, BG] etc). Let us give first a brief review of those studies.

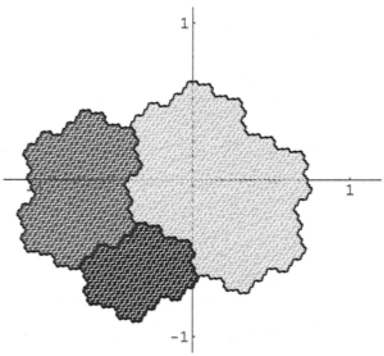

Figure 1. Atomic surfaces $\bigcup_{i=1,2,3} X_i$ of Rauzy substitution.

Fractal geometry and self-similar tilings:

Luck et al. ([Luc]) studied the atomic surfaces from the point of view of physics. They observed that the atomic surfaces have self-similar structures and usually have fractal boundaries.

Arnoux and Ito ([AI]) and Sirvent and Wang ([SW]) established many elementary facts concerning the atomic surfaces. They deduced the set equations of partial atomic surfaces, so that the partial atomic surfaces are proved to be the invariant sets of a graph-IFS, and the graph-IFS satisfies an **open set condition** (see [MW] [Fa2] for details concerning graph-IFS and the open set condition). Arnoux and Ito introduced the notions of stepped-surface and dual substitution, that appear to be suitable to handle the graph-directed structure.

A graph-IFS, with open set condition and having invariant sets with non-empty interiors, is interesting in tiling theory (see [Vin] and references therein). The equations of partial atomic surfaces $\{X_i\}_{1\leq i\leq d}$ provide a dilation and subdivision rule, and hence give us a self-similar tiling system. The sets $\{X_i\}_{1\leq i\leq d}$ can tile the space by translation. These tilings have several prototiles and they are quasi-periodic ([SW]). They may serve as models of quasi-crystals. However, in the present paper, our main concern is to study a particular and important collection different from the above tilings.

The dimensions of the boundaries of atomic surfaces are studied by Feng et al. [FF].

Spectrum, domain exchange and coincidence:

A motivation of Rauzy's construction is to study in detail the spectral type of **substitutive dynamical system**, that is, the dynamical system defined as the shift map acting on symbolic infinite sequences that have the same language as any periodic point of the substitution. It is well known that a dynamical system has pure discrete spectrum if and only if it is metrically isomorphic to a translation on a compact abelian group, and a substitutive system has a pure discrete spectrum if and only if it is semi-topologically conjugated to a translation on a compact group (see [Hos1] [Que] [Fog]).

For an irreducible and unimodular Pisot substitution, Arnoux and Ito [AI] defined a **domain exchange transformation** on the atomic surface X, which is metrically conjugate to the substitutive dynamical system provided a **strong coincidence condition** holds. This condition was first introduced for substitutions of constant length by Dekking [Dek] who proved that it characterizes systems with pure discrete spectrum. [AI] generalized the strong coincidence condition to the non-constant length case. However, a metric conjugacy to a domain exchange does not imply that there exists a metric conjugacy to a translation on an abelian group. A sufficient condition is that there exists a lattice Γ such that the collection $\mathcal{J}_1 = \{X + u\colon u \in \Gamma\}$ defines a tiling of the space. (See Figure 2.) Siegel [Sie1] and Thuswaldner [Thu] define some graph combinatorial conditions (derived from arithmetic properties) for $\mathcal{J}_1$ to be a tiling.

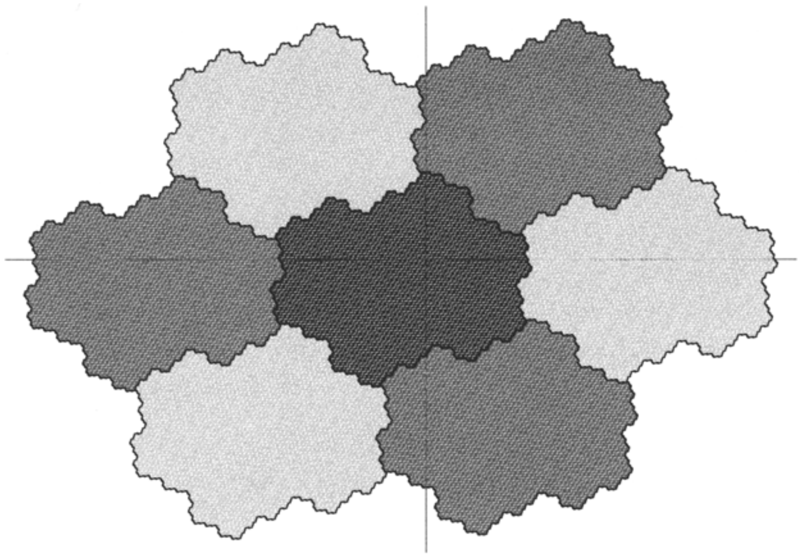

Figure 2. The collection $\mathcal{J}_1$ of Rauzy substitution.

For any unimodular Pisot substitution σ over two letters (*Pisot substitution over two letters must be irreducible if we agree that rational numbers are not Pisot*), Host [Hos2] showed that $\mathcal{J}_1$ is a tiling if σ satisfies the strong coincidence condition. Barge and Diamond [BD] proved that the strong coincidence condition holds for any Pisot substitution over two letters. Hence a substitutive dynamical system has pure discrete spectrum provided that the substitution is a unimodular Pisot substitution over two letters. Using a different approach, Hollander and Solomyak [HS] improved this result by removing the unimodular assumption.

Markov partition and purely periodic β-expansion:

If the substitution σ is unimodular, then the incidence matrix M_σ defines a group automorphism on the torus. The atomic surfaces are employed to construct Markov partitions of M_σ in [Pra,IO, Sie2, KV]. They build a d-dimensional domain $\hat{X} = \bigcup_{i=1}^{d} \hat{X}_i$, which is a Markov partition of the group automorphism M_σ as soon as $\hat{X}$ is a fundamental domain of $\mathbb{Z}^d$, that is, as soon as the collection $\mathcal{J}_2 = \{\hat{X} + z\colon z \in \mathbb{Z}^d\}$ is a tiling. (See Figure 3.) Praggastis [Pra] proves that $\hat{X}$ generates a Markov partition provided that the characteristic polynomial of the substitution satisfies the (F)-condition as defined by

Frougny and Solomyak [FS].

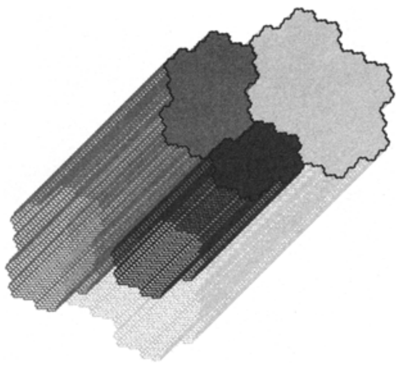

Figure 3. $\widehat{X}$ of Rauzy substitution.

Thurston [T] constructed self-similar tilings from a β-numeration system when β is a Pisot number and an algebraic unit. This β-tiling system has been studied in detail by Akiyama ([Ak1] [Ak2]). The construction of Thurston appears to be a special case of atomic surfaces as defined in [EI, EIR] for unimodular Pisot substitutions which are not necessarily irreducible. Ito and Sano [IS] and Ito and Rao [IR] found the connection between the domain $\hat{X}$ and the arithmetic properties of the β-expansion.

1.2. Main results. The class of irreducible and unimodular Pisot substitutions will be defined in Section 2.1. Several types of tilings have been associated to this class of substitutions. In the present paper, we construct a new quasi-periodic collection $\mathcal{J}$, which consists of translations of partial atomic surfaces. (See Figure 4.) Our main concern is to investigate when $\mathcal{J}$ is a tiling. The collection $\mathcal{J}$ is important since:

(i) It contains the β-tiling studied by Thurston [T] and Akiyama [Ak1, Ak2] as a subclass, and thus unifies the study of atomic surface tilings and β-tilings.

(ii) We shall show that the collections $\mathcal{J}$, $\mathcal{J}_1$ and $\mathcal{J}_2$ either are all tilings, or are all not tilings. Hence $\mathcal{J}$ is closely related to dynamical systems on atomic surfaces.

(iii) We shall see that $\mathcal{J}$ is a tiling if and only if σ satisfies a combinatorial condition called the **super-coincidence condition**. Hence $\mathcal{J}$ plays an interrole between tilings, dynamical systems, coincidence and spectral properties.

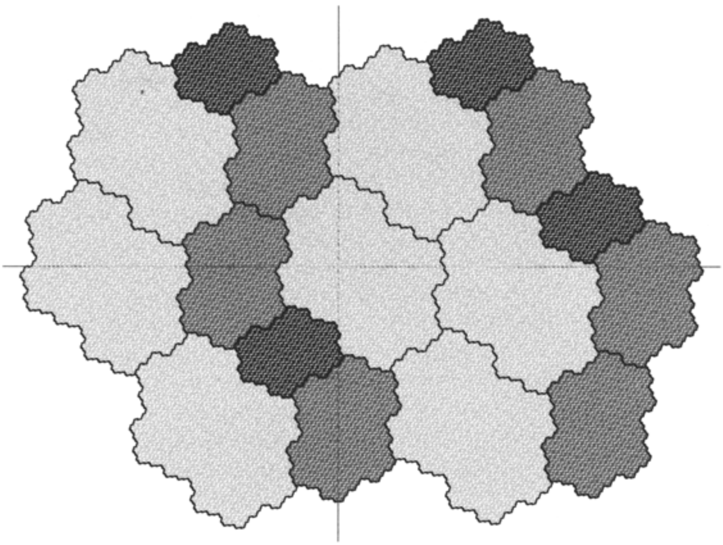

Figure 4. The collection $\mathcal{J}$ of the Rauzy substitution.

Let us give more details on the collection $\mathcal{J}$ and the results we will prove in this paper. By the definition of Pisot substitution, the linear map of $\mathbb{R}^d$ defined by the incidence matrix M_σ has a stable space P of dimension $d-1$ and an unstable space of dimension 1. A stepped surface is associated to the stable hyperplane P, defined as the discrete plane that provides the best approximation of P. By projecting this stepped-surface on P, we obtain a polygonal tiling of P consisting of d types of polygons. Replacing the polygons of type i by the partial atomic surface X_i, we obtain the collection $\mathcal{J}$ (see Section 3).

We first show that $\mathcal{J}$ is a **quasi-periodic** collection and has a **self-replicating** property. Combining these two properties together, we prove that P is equally covered by $\mathcal{J}$.

THEOREM 1.1: *There exists a constant k, such that almost every point of the stable space P is covered by k pieces of tiles in $\mathcal{J}$.*

It is natural to ask when $\mathcal{J}$ is a tiling. A number of equivalent conditions are given in Theorem 3.3. This theorem is an analogue to Theorem 4.3 of Vince [Vin], dealing with another class of self-similar tilings, called **integral self-affine tilings**. The periodic collection of atomic surfaces $\mathcal{J}_1$ was previously introduced, related to a domain exchange transformation; so was the collection

$\mathcal{J}_2$, related to a potential Markov partition for a toral automorphism. In Section 3, we prove the following.

THEOREM 1.2: *The three collections $\mathcal{J}$, $\mathcal{J}_1$ and $\mathcal{J}_2$ are tilings simultaneously provided one of them is a tiling.*

Can we check whether $\mathcal{J}$ is a tiling by just looking at the substitution? To answer this question, in Section 4, we introduce a combinatorial notion of super-coincidence condition, that implies the strong coincidence condition. As we shall see, the super-coincidence is the strongest coincidence condition one can have.

THEOREM 1.3: *The collection $\mathcal{J}$ is a tiling if and only if σ satisfies the super-coincidence condition.*

The well known *Coincidence Conjecture* says that any irreducible Pisot substitution satisfies the strong coincidence condition. Here we boldly conjecture that every irreducible Pisot substitution satisfies the super-coincidence condition. We show that this conjecture is true for Pisot substitutions over two letters, thanks to the major advance made by Barge and Diamond [BD].

THEOREM 1.4: *Pisot substitutions (not necessarily unit) over two letters satisfy the super-coincidence condition.*

In [BR], we carry out more detailed studies on the super-coincidence condition. We show that if σ satisfies this condition, then σ is of Pisot or Salem type, and the associated symbolic dynamical system has a pure discrete spectrum. The atomic surfaces of reducible Pisot substitutions are studied in a subsequent paper [EIR].

The paper is organized as follows: In Section 2, we establish basic notations and definitions, including atomic surfaces, stepped-surface and dual substitution. In Section 3, we define the collection $\mathcal{J}$ and study the relations between $\mathcal{J}$, $\mathcal{J}_1$ and $\mathcal{J}_2$. Theorem 1.1 and Theorem 1.2 are proved in Section 3. Section 4 is devoted to the super-coincidence condition. Theorem 1.3 and Theorem 1.4 are proved there.

2. Atomic surface, stepped-surface and dual substitution

2.1. DEFINITIONS AND NOTATIONS. Let $\mathcal{A} = \{1, \ldots, d\}$ be an alphabet, $d \geq 2$. Let $\mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n$ be the set of finite words. A substitution is a function $\sigma\colon \mathcal{A} \mapsto \mathcal{A}^*$. The incidence matrix of σ is $M_\sigma = M = (m_{ij})_{1\leq i,j\leq d}$, where m_{ij} is the number of occurrences of i in $\sigma(j)$. A matrix A is **primitive** if there exists a positive integer N such that A^N is a positive matrix. We will always assume that the incidence matrix M is primitive. An infinite word ω is a **fixed point** of σ if $\sigma(\omega)=\omega$; it is a **periodic point** if $\sigma^k(\omega)=\omega$ for some integer $k\geq 1$. A primitive substitution has at least one periodic point.

A substitution σ is said to be **a Pisot substitution** if the Perron–Frobenius eigenvalue of M is a Pisot number (an algebraic integer is **a Pisot number** if all its algebraic conjugates have modulus strictly less than 1). A substitution σ is **unimodular** if $\det M=\pm 1$; it is **irreducible** if the characteristic polynomial of M is irreducible over $\mathbb{Q}$. It is shown that an irreducible Pisot substitution is always primitive ([HZ]).

The following notations are used throughout this paper. Denote by $e_1,\dots,e_d$ the canonical basis of $\mathbb{R}^d$. Define a map $f\colon \mathcal{A}^*\to\mathbb{Z}^d$ by

(i) $f(\epsilon)=0$, where ϵ denotes the empty word;

(ii) $f(i)=e_i$ for $1\leq i\leq d$ and $f(UV)=f(U)+f(V)$ for $U,V\in\mathcal{A}^*$.

We write $\sigma(i)=W_1^{(i)}\cdots W_{l_i}^{(i)}$, $i=1,2,\dots,d$. We denote by

$$P_k^{(i)}=W_1^{(i)}\cdots W_{k-1}^{(i)}$$

the prefix of $\sigma(i)$ with length $k-1$, and denote by

$$S_k^{(i)}=W_{k+1}^{(i)}\cdots W_{l_i}^{(i)}$$

the suffix of $\sigma(i)$. Then $\sigma(i)$ can be written as

$$\sigma(i)=P_k^{(i)}W_k^{(i)}S_k^{(i)}.$$

For $x\in\mathbb{Z}^d$, we denote by $(x,i):=\{x+\theta e_i;\ \theta\in[0,1]\}$ the segment from x to $x+e_i$. Define $y+(x,i):=(x+y,i)$. Let $\mathcal{G}=\{(x,i)\colon x\in\mathbb{Z}^d, 1\leq i\leq d\}$. The term *segment* will refer to any element of $\mathcal{G}$.

The **inflation and substitution map** F_σ is defined as follows on the set of subsets of $\mathcal{G}$. Define

$$F_\sigma(0,i):=\bigcup_{k=1}^{l_i}\{(f(P_k^{(i)}),W_k^{(i)})\},\quad 1\leq i\leq d.$$

$$F_\sigma(x,i):=\{Mx+\mathbf{k}\colon \mathbf{k}\in F_\sigma(0,i)\},$$

and for $K\subseteq\mathcal{G}$,

$$F_\sigma(K):=\{F_\sigma(\mathbf{k})\colon \mathbf{k}\in K\}.$$

Rigorously we should use $F_\sigma\{(x,i)\}$ instead of $F_\sigma(x,i)$. By the definitions of the incidence matrix and f, one has $M_{\sigma^n}=M_\sigma^n$ and $f(\sigma(U))=M_\sigma f(U)$ for every word U, and so that $F_{\sigma^n}=F_\sigma^n$.

For a finite or infinite word $s = s_1 \cdots s_n \ldots$, a broken line $\overline{s}$ starting from the origin is defined as follows:

$$\overline{s} = \bigcup_{i \geq 1} \{(f(s_1 \cdots s_{i-1}), s_i)\}.$$

2.2. ATOMIC SURFACE. Let σ be an irreducible and unimodular Pisot substitution. Let λ be the Perron–Frobenius eigenvalue of M. Then the linear map of $\mathbb{R}^d$ defined by the incidence matrix M has a stable space P of dimension $d-1$ and an unstable space of dimension 1 spanned by a positive Perron–Frobenius eigenvector v. According to the direct sum $\mathbb{R}^d = V \oplus P$, we define two natural projections

$$\pi\colon \mathbb{R}^d \mapsto P, \ \pi'\colon \mathbb{R}^d \mapsto V.$$

Since the Perron–Frobenius eigenvector is non-rational, the projection on the lattice $\mathbb{Z}^d$ is dense in P as well as V; it is also one-to-one on $\mathbb{Z}^d$: for any $x, y \in \mathbb{Z}^d$, $\pi(x) = \pi(y)$ (or $\pi'(x) = \pi'(y)$) if and only if $x = y$.

Let $\omega = s_1 s_2 \cdots$ be a periodic point of σ. Let $Y = \{f(s_1 \cdots s_{k-1})\colon k \geq 1\}$ be the set of integer points located on $\overline{\omega}$. The **atomic surface** of σ is the closure $X = \overline{\pi(Y)}$ of the projection of Y onto the stable space P. Furthermore, let $Y_i = \{f(s_1 \cdots s_{k-1})\colon s_k = i, \ k \geq 1\}$, and $X_i = \overline{\pi(Y_i)}$, $1 \leq i \leq d$. We call the family $\{X_i\}_{1 \leq i \leq d}$ **the partial atomic surfaces** of σ. Clearly $X = \bigcup_{i=1}^d X_i$.

In Theorem 2.1, some well-known properties of atomic surfaces are listed.

THEOREM 2.1: *Let σ be an irreducible and unimodular Pisot substitution. Then:*

(i) *The partial atomic surfaces $\{X_i\}_{i=1}^d$ are compact and satisfy the following set equations,*

$$M^{-1} X_i = \bigcup_{j=1}^{d} \bigcup_{W_k^{(j)} = i} (X_j + M^{-1} \pi(f(P_k^{(j)}))). \tag{2.1}$$

(ii) *The interior of X_i is not empty and $X_i = \overline{X_i^\circ}$.*

(iii) $\dim_H \partial X_i \leq \dim_B \partial X_i < d-1$, *where* $\dim_H$ *and* $\dim_B$ *denote respectively the Hausdorff dimension and Box-counting dimension.*

(iv) *The right side of (2.1) consists in non-overlapping unions.*

Items (i), (ii) and (iv) are proved by [AI] (inspired by an unpublished paper [Hos2]) and can be found in [SW]; item (iii) is proved by [FF].

Remark 2.2: (1) Since the invariant sets satisfying (2.1) are unique (see [Fa2] [MW]), the atomic surfaces do not depend on the choice of the periodic point ω.

(2) As a consequence of (iii), the $(d-1)$-dimensional Lebesgue measure of the boundaries of X_i are zero.

2.3. STEPPED-SURFACE. The stepped-surface and the dual substitution associated with a Pisot substitution are introduced by Arnoux and Ito [AI]; they play an important role in the present paper.

Let P be a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. Let v^* be a vector which is orthogonal to P and has Euclidean norm 1. The vector v^* may have two different values. When P stands for the stable space determined by an irreducible Pisot substitution, v^* is chosen to be the Perron–Frobenius eigenvector of ${}^tM$ with Euclidean norm 1 , where ${}^tM$ denotes the transpose of M.

Denote by $\langle\cdot,\cdot\rangle$ the inner product of the Euclidean space $\mathbb{R}^d$. Let

$$P^+ := \{x \in \mathbb{R}^d \colon \langle x, v^*\rangle \geq 0\} \quad \text{and} \quad P^- := \{x \in \mathbb{R}^d \colon \langle x, v^*\rangle < 0\}.$$

From its definition, P^+ denotes the half space above the subspace P, according to the direction given by v^*.

We now intend to associate to any integer point $\mathbb{Z}^d$ a color described by a letter on $\mathcal{A}$. Formally, let us denote

$$\mathcal{G}^* := \{[x, i^*] \colon x \in \mathbb{Z}^d, 1 \leq i \leq d\}$$

the set of such colored points. We formally define S as the set of *nearest* colored points $[x, i^*]$ above P, meaning that x belongs to P^+ whereas $x - e_i$ doesn't.

$$S := \{[x, i^*] \in \mathcal{G}^* \colon x \in P^+ \text{ and } x - e_i \in P^-\}. \tag{2.2}$$

Notice that an integer x may have several colors.

To any colored integer point $[x, i^*]$, one associates a face of a unit cube of $\mathbb{R}^d$ whose lowest vertex is in x and that is orthogonal to the direction e_i. We denote this face by $\overline{[x, i^*]}$, namely

$$\overline{[x, i^*]} := \{x + c_1 e_1 + \cdots + c_i e_i + \cdots + c_d e_d \in \mathbb{R}^d \colon c_i = 0,\ 0 \leq c_k \leq 1 \text{ if } k \neq i\}.$$

Then the **stepped-surface** of P is the surface $\bar{S}$ defined by

$$\overline{S} := \bigcup_{[x,i^*] \in S} \overline{[x, i^*]}.$$

By an abuse of language, the formal set S will also be called the stepped-surface of P.

For $u \in \mathbb{R}^d$, denote by $P+u$ the translation of P. That is, $P+u$ is a hyperplane which is parallel to P and contains u. The space between P and $P+u$, including P and excluding $P+u$, is denoted by $[P, P+u)$. As a consequence of the definition of S, $[x, i^*] \in S$ if and only if $x \in [P, P+e_i)$.

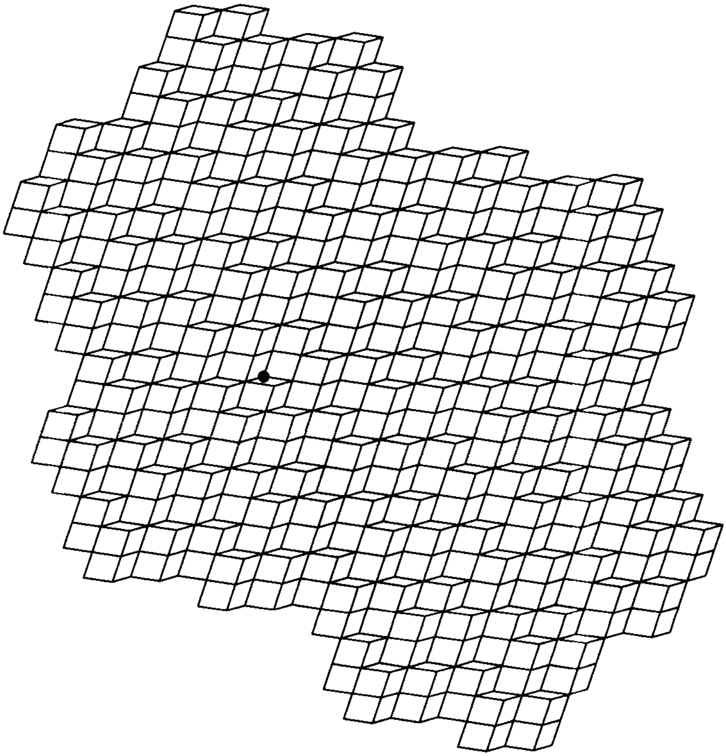

Figure 5. The stepped-surface of Rauzy substitution.

We will denote $\overline{[i^*]} = \overline{[0, i^*]}$ the faces of the unit cube placed in zero. Projecting the stepped-surface $\overline{S}$ to P, we obtain a polygonal tiling of P,

$$\mathcal{J}' := \{\pi\overline{[x, i^*]}: [x, i^*] \in S\}, \tag{2.3}$$

which uses d polygons $\pi\overline{[1^*]}, \ldots, \pi\overline{[d^*]}$ as prototiles.

Denote by $\mu(\cdot)$ the $(d-1)$-dimensional Lebesgue measure; then the vector

$$(\mu(X_1), \ldots, \mu(X_d))$$

is a Perron–Frobenius eigenvector of M (see [AI] or [SW]). It is easy to show that

LEMMA 2.3: *The vector $(\mu(\pi\overline{[1^*]}),\dots,\mu(\pi\overline{[d^*]}))$ is a Perron–Frobenius eigenvector of the incidence matrix M.*

Proof: The parallelogram $\pi\overline{[i^*]}$ is generated by the vectors $\pi(e_j)$ with $j\neq i$. It is orthogonal to the normalized vector v^*. Hence, its measure is equal to

$$|\det(v^*,\pi(e_1),\dots,\widehat{\pi(e_i)},\dots,\pi(e_d))|,$$

where $\widehat{\pi(e_i)}$ means that the term $\pi(e_i)$ is omitted. Since v^* is orthogonal to the space P, one has

$$x=\pi(x)+\frac{{}^tv^*x}{{}^tv^*v}v$$

for all $x\in\mathbb{R}^d$ so that

$$\begin{aligned}|\det(v^*,\pi(e_1),\dots,\widehat{\pi(e_i)},\dots,\pi(e_d))| &=\frac{{}^tv^*v^*}{{}^tv^*v}|\det(v,\pi(e_1),\dots,\widehat{\pi(e_i)},\dots,\pi(e_d))|\\ &=\frac{1}{{}^tv^*v}|\det(v,e_1,\dots,\widehat{e_i},\dots,e_d)|\\ &=\frac{{}^tve_i}{{}^tv^*v}.\end{aligned}$$

Hence $(\mu(\pi\overline{[1^*]}),\dots,\mu(\pi\overline{[d^*]}))=(1/{}^tv^*v){}^tv$ is a Perron–Frobenius eigenvector of M. ∎

2.4. DUAL SUBSTITUTIONS. For $[y,i^*]\in\mathcal{G}^*$, we define $x+[y,i^*]:=[x+y,i^*]$. According to set equations (2.1), we define the **dual substitution** σ^* for every subset of $\mathcal{G}^*$ as follows:

$$\sigma^*[0,i^*]:=\bigcup_{j=1}^{d}\bigcup_{W_k^{(j)}=i}\{[M^{-1}f(P_k^{(j)}),j^*]\},\ 1\le i\le d,$$

$$\sigma^*[x,i^*]:=\{M^{-1}x+\mathbf{k}\colon \mathbf{k}\in\sigma^*[0,i^*]\},$$

and for any $K\subseteq\mathcal{G}^*$,

$$\sigma^*(K):=\{\sigma^*(\mathbf{k})\colon \mathbf{k}\in K\}.$$

For simplicity, we use $\sigma^*[x,i^*]$ to denote $\sigma^*\{[x,i^*]\}$. Using the dual substitution, formula (2.1) is equivalent to

$$M^{-1}X_i=\bigcup_{[x,k^*]\in\sigma^*[0,i]}\pi(x)+X_k, \tag{2.4}$$

and the right side is a non-overlapping union. It is easy to see that $(\sigma^*)^n = (\sigma^n)^*$ (see [AI]). From these we deduce

THEOREM 2.4: *Let σ be an irreducible and unimodular Pisot substitution. If $[x, i^*]$ and $[y, j^*]$ belong to $(\sigma^*)^n[0, k^*]$ for some n and k, then $\pi(x) + X_i$ and $\pi(y) + X_j$ do not overlap.*

Proof: Set $\tau = \sigma^n$; then τ is also an irreducible and unimodular Pisot substitution. The incidence matrix of τ is M^n and the dual substitution is $\tau^* = (\sigma^*)^n$. Hence by (2.4),

$$M^{-n}X_k = \bigcup_{[x,i^*]\in(\sigma^*)^n[0,k]} \pi(x) + X_i,$$

and the right side is a non-overlapping union. Therefore, $\pi(x)+X_i$ and $\pi(y)+X_j$ do not overlap since both of them belong to this non-overlapping union. ∎

A remarkable property of the dual substitution is that it keeps the stepped-surface invariant ([AI]).

THEOREM 2.5 ([AI]): *If σ is an irreducible and unimodular Pisot substitution, then the stepped-surface is invariant under the action of the dual substitution σ^*. Precisely,*

(i) $\sigma^*(S) = S$.

(ii) $\sigma^*[x, i^*] \cap \sigma^*[y, j^*] = \emptyset$, *for* $[x, i^*], [y, j^*] \in S$ *with* $[x, i^*] \neq [y, j^*]$.

2.5. QUASI-PERIODICITY OF THE STEPPED-SURFACE. Let $\mathcal{J}$ be a translation tiling of $\mathbb{R}^d$ with finite many prototiles. Denote by $B(x, r)$ the ball centered on x and with radius r. We call

$$\mathcal{P}(B(x,r)) = \{T : T \in \mathcal{J} \text{ and } T \cap B(x,r) \neq \emptyset\}$$

the **local arrangement** in $B(x, r)$.

If, for any $r > 0$, there exists $R > 0$ such that for any $x, y \in \mathbb{R}^n$, the local arrangement of $B(x, r)$ appears (up to translation) in the ball $B(y, R)$, then we say that $\mathcal{J}$ is **quasi-periodic**.

Now we define the quasi-periodicity for a stepped-surface. Let S be the stepped-surface of a $(d-1)$-dimensional subspace P. Let

$$S_0 = \{x : [x, i^*] \in S\}$$

be the **base set** of S. For $z \in S_0$, we define the **local arrangement** of a ball $B(z, r)$ in S as

$$\mathcal{P}(B(z,r)) = \{[x, i^*] : [x, i^*] \in S \text{ and } x \in B(z,r)\}.$$

Definition 2.6: A stepped-surface S is said to be quasi-periodic provided it satisfies the following property: for any $r > 0$, there exists $R > 0$ such that for any $x, y \in S_0$, the local arrangement of $B(x, r)$ appears (up to translation) in the local arrangement of the ball $B(y, R)$ in S.

[ABI] proved the quasi-periodicity of the stepped-surface by associating the stepped-surface with a certain dynamical system on the real line.

PROPOSITION 2.7 ([ABI]): *Let P be a subspace of $\mathbb{R}^d$ with dimension $d-1$. Then the corresponding stepped-surface S is quasi-periodic.*

COROLLARY 2.8: *The set $\pi(S_0)$ is uniformly discrete in P, i.e., there is a constant $r > 0$ such that $|x - y| > r$ holds for any $x, y \in \pi(S_0)$.*

3. Three tilings related to atomic surfaces

3.1. THE COLLECTION $\mathcal{J}$. Let us recall that $\mathcal{J}'$ is the polygonal tiling of P defined in Section 2. Replacing the polygons $\pi(x) + \pi[i^*]$ in $\mathcal{J}'$ by the fractal sets $\pi(x) + X_i$, we get a collection (see Figure 4)

$$\mathcal{J}_\sigma =: \mathcal{J} = \{\pi(x) + X_i \colon [x, i^*] \in S\}. \tag{3.1}$$

The collection $\mathcal{J}$ is self-replicating. The set

$$M^{-1}\mathcal{J} := \{M^{-1}T : T \in \mathcal{J}\}$$

is a collection with the prototiles $\{M^{-1}X_i\}_{i=1}^d$. Subdividing $M^{-1}X_i$ into small pieces according to the set equations (2.1), we obtain a new collection. By Theorem 2.5, the new collection coincides with $\mathcal{J}$, so that $\mathcal{J}$ satisfies the *self-replicating property* defined by Kenyon [Ken].

The collection $\mathcal{J}$ is quasi-periodic. This follows immediately from the quasi-periodicity of the stepped-surface.

Example 3.1: Let β be a Pisot number and an algebraic unit with the minimal polynomial $x^d = a_1x^{d-1} + a_2x^{d-2} + \cdots + a_d$, where $a_i \geq 0$ for $1 \leq i \leq d$. Let us assume that the β-expansion of 1 is $0.a_1a_2\cdots a_d$. Then the β-numeration system is governed by the following substitution σ over $\{1, \ldots, d\}$:

$$\begin{aligned} 1 &\mapsto 1^{a_1}2, \\ 2 &\mapsto 1^{a_2}3, \\ &\cdots \\ \text{d-1} &\mapsto 1^{a_{d-1}}\text{d}, \\ \text{d} &\mapsto 1^{a_d}. \end{aligned}$$

Here 1^n denotes the word consisting of n consecutive 1's. It is shown in [EIR] that $\mathcal{J}$ is a refinement of the collection constructed by Thurston [T]. Akiyama [Ak1, Ak2] proved that the collection of Thurston, and thus $\mathcal{J}$, is a tiling provided that $a_1 \geq a_2 \geq \cdots \geq a_d$.

Let us first show that P is equally covered by $\mathcal{J}$.

THEOREM 1.1: *There exists a constant k, such that almost every point of the stable space P is covered by k pieces of tiles in $\mathcal{J}$.*

Proof: For $x \in P$, we define the covering degree of x to be

$$D(x) := \sharp\{T \in \mathcal{J} \colon x \in T\}.$$

The covering degree is bounded since X_i are bounded and the set $\pi(S_0)$ is uniformly discrete.

Let $B = \bigcup_{T\in\mathcal{J}} \partial T$ denote the union of boundaries of tiles in the covering $\mathcal{J}$. This set is closed since it is the union of closed sets that are locally finite. It also has a zero measure since the boundary of each X_i has measure 0 (see Remark 2.2). Then $P \setminus B$ is an open set, so that for every $x_0 \in P \setminus B$, there exists $B(x_0, r) \subset P \setminus B$. The definition of the boundary implies that as soon as $B(x_0, r)$ intersects a tile $T \in \mathcal{J}$, then it is fully contained in the tile: if $B(x_0, r) \cap T \neq \emptyset$ and $B(x_0, r) \cap (P \setminus T) \neq \emptyset$, by an argument of connectivity, $B(x_0, r)$ has to cross the boundary, that is, $B(x_0, r) \cap \partial T \neq \emptyset$, which is impossible since $B(x_0, r) \cap B = \emptyset$. Consequently, the covering degree $D(x)$ is constant on the ball $B(x_0, r)$.

Since $\mathcal{J}$ is a self-replicating collection, $D(x) = k$ in a ball $B(x_0, r)$ implies that $D(x) = k$ holds almost everywhere in every domain $M^{-n}B(x_0, r)$ for $n \geq 0$. Let $D(x) = k$ be the degree on a ball $B(x_0, r)$ with $x_0 \in P \setminus B$, and $D(x) = k'$ on another ball $B(y_0, r)$ with $y_0 \in P \setminus B$. For N large enough, $M^{-N}B(x_0, r)$ contains an arbitrarily large ball that must contain a translation of the local arrangement of $B(y_0, r)$ by the quasi-periodicity of $\mathcal{J}$. This implies that $k = k'$. As a conclusion, the covering degree is constant on the set $P \setminus B$. ∎

It is natural to ask when $\mathcal{J}$ is a tiling. Theorem 3.3 gives some equivalent conditions: it should be considered as an analogue to Theorem 4.3 in [Vin], which deals with integral self-affine tilings.

Let E be a subset of $\mathbb{R}^d$. An element $[x, i^*]$ is said to have a base point in E if $x \in E$. We denote by S_E the subset of the stepped-surface which has base points in E. Let

$$X_i(n) := M^n \bigcup_{[z,j^*]\in(\sigma^*)^n[0,i^*]} \pi\overline{[z,j^*]}. \tag{3.2}$$

Let us embed the set of compact subsets of P with the Hausdorff metric D_H, that is, $\lim_{n\to\infty} A_n = A$ if and only if $\lim_{n\to\infty} D_H(A_n, A) = 0$. It was proved in [AI] [SW] that the polygonal sets $X_i(n)$ converge to X_i with respect to the Hausdorff metric.

PROPOSITION 3.2: $\lim_{n\to\infty} X_i(n) = X_i$.

In the following theorem, we introduce five equivalent conditions for tilings, that are related to the polygonal pieces $X_i(n)$.

THEOREM 3.3: *The following statements are equivalent:*

(i) *The collection $\mathcal{J}$ is a tiling of P.*
(ii) *For some (or for all) i, $\mu(X_i) = \mu(\pi\overline{[i^*]})$, where $\pi\overline{[i^*]}$ denotes polygonal tiles of the tiling $\mathcal{J}'$.*
(iii) *For some (or for all) i, the radius of the largest ball B with $S_B \subseteq (\sigma^*)^n[0,i^*]$ tends to infinity when n tends to infinity.*
(iv) *For some (or for all) i, $\lim \partial X_i(n) = \partial X_i$.*
(v) *For some (or for all) i, $\lim \partial X_i(n) \neq X_i$.*

Proof: In this proof, $B(a,R)$ will always denote a ball in P.

(i)⇔(ii). Since both $(\mu(X_1),\dots,\mu(X_d))$ and $(\mu(\pi\overline{[1^*]}),\dots,\mu(\pi\overline{[d^*]}))$ are Perron–Frobenius eigenvectors of M, there exists a real number k such that

$$(\mu(X_1),\dots,\mu(X_d)) = k(\mu(\pi\overline{[1^*]}),\dots,\mu(\pi\overline{[d^*]})). \tag{3.3}$$

Suppose that $\mathcal{J}$ covers P with multiplicity k' as defined in Theorem 1.1; we claim that $k = k'$.

Since $\{\pi\overline{[x,j^*]}: x \in B(0,R)\}$ is a part of the polygonal tiling $\mathcal{J}'$, we deduce

$$\frac{\sum_{x\in B(0,R)} \mu(\pi\overline{[x,j^*]})}{\mu(B(0,R))} \to 1 \quad (R\to\infty). \tag{3.4}$$

On the other hand, since $\mathcal{J}$ is a covering of P of degree k', we have

$$\frac{\sum_{x\in B(0,R)} \mu(\pi(x)+X_j)}{\mu(B(0,R))} \to k' \quad (R\to\infty). \tag{3.5}$$

Formulas (3.3), (3.4) and (3.5) imply that $k = k'$. Hence (i)⇔ (ii).

(i)⇔(iii). Let $B(z_n, R_n)$ be the largest ball contained in $M^{-n}X_i$.

If $\mathcal{J}$ is a tiling, then it is a self-replicating tiling. Thus for any i, $1 \le i \le d$,

$$\{\pi(x) + X_j\colon [x, j^*] \in (\sigma^*)^n[0, i^*]\} \tag{3.6}$$

is a patch of the tiling $\mathcal{J}$. It tiles $M^{-n}X_i$ so that it covers the ball $B(z_n, R_n)$. For any $[y, j^*] \notin (\sigma^*)^n[0, i^*]$, $\pi(y) + X_j$ does not belong to the above patch. Hence $(\pi(y) + X_j) \cap B(z_n, R_n) = \emptyset$ and thus $\pi(y) \notin B(z_n, R_n - C_1)$, where C_1 is a real such that $\bigcup_{1\le k\le d} X_k \subseteq B(0, C_1)$. Therefore the ball $S_{B(z_n, R_n - C_1)} \subseteq (\sigma^*)^n[0, i^*]$, and $R_n - C_1 \to \infty$ when $n \to \infty$. So (i)⇒(iii).

Suppose that (iii) holds, meaning that there exists a sequence of balls $B(z_n, R_n)$ such that $S_{B(z_n,R_n)} \subseteq (\sigma^*)^n[0, i^*]$ and $R_n \to \infty$. By Theorem 2.4, the patch (3.6) is non-overlapping, hence

$$\overline{\lim}_{R_n\to\infty} \frac{\sum_{x\in B(z_n,R_n)} \mu(\pi(x) + X_j)}{\mu(B(z_n, R_n))} \le 1. \tag{3.7}$$

But by (3.5), the left side of (3.7) is equal to k'. Hence $k' = 1$ and $\mathcal{J}$ is a tiling.

(i)⇔(iv). Assume that $\mathcal{J}$ is a tiling. Let

$$C_2 = \operatorname{diam}\left(\bigcup_{i=1}^{d} X_i\right) + \operatorname{diam}\left(\bigcup_{i=1}^{d} \pi\overline{[i^*]}\right).$$

Notice that $M^{-n}X_i$ (after subdivision) is a patch of the tiling $\mathcal{J}$ and $M^{-n}X_i(n)$ (after subdivision) is a patch of the tiling $\mathcal{J}'$. Since the translation sets of these two tilings coincide, $B(z, R) \subseteq M^{-n}X_i$ implies that $B(z, R - C_2) \subseteq M^{-n}X_i(n)$, and on the other side $B(z, R) \subseteq M^{-n}X_i$ implies $B(z, R - C_2) \subseteq M^{-n}X_i(n)$. This implies that

$$D_H(\partial(M^{-n}X_i), \partial(M^{-n}X_i(n))) < 2C_2.$$

Hence (iv) holds.

Assume that (iv) is true. From the assumption and the fact that $X_i(n)$ tends to X_i, for any $\epsilon > 0$, there exists an integer $n = n(\epsilon)$ such that

$$D_H(X_i, X_i(n)) < \epsilon, \quad D_H(\partial X_i, \partial X_i(n)) < \epsilon. \tag{3.8}$$

By the first inequality, for any $y \in X_i \setminus X_i(n)$, there exists a point $y' \in B(y, \epsilon) \cap X_i(n)$. Hence the line segment from y to y' must intersect the boundary of $X_i(n)$. Therefore $|y - y''| < \epsilon$ for some $y'' \in \partial X_i(n)$, implying that

$$X_i \subseteq X_i(n) \cup [\partial X_i(n)]_\epsilon,$$

where $[E]_\epsilon := \{x: |x-y| < \epsilon \text{ for some } y \in E\}$. Since $\mu(X_i(n)) = \mu(\pi[i^*])$ and $\mu(X_i)/\mu(\pi[i^*]) = k' > 1$, the condition (ii), as well as condition (i), will be satisfied as soon as the following holds:

$$\lim_{\epsilon\to 0} \mu([X_i(n)]_\epsilon) = 0. \tag{3.9}$$

The second inequality of (3.8) implies that $[\partial X_i(n)]_\epsilon \subseteq [\partial X_i]_{2\epsilon}$. But $\dim_B \partial X_i < d-1$ implies that $\lim_{\epsilon\to 0}\mu([\partial X_i]_{2\epsilon}) = 0$ (see [Fal]). So (3.9) holds.

(iii)$\Leftrightarrow$(v). The implications (iii)$\Rightarrow$(i)$\Rightarrow$(iv)$\Rightarrow$(v) are trivial.

Conversely, suppose that (iii) is false. There exists a constant C_3 such that for any x and any n, $S_{B(x,C_3)}$ contains an element $[z, j^*]$ which does not belong to $(\sigma^*)^n[0,i^*]$. Choose x to be an arbitrary point in $X_i(n)$. Then $\pi(z)$ does not belong to the interior of $X_i(n)$ and $|x-\pi(z)| < C_3$. So we conclude that

$$\begin{gathered} D_H(M^{-n}X_i(n), \partial(M^{-n}X_i(n))) < C_3, \\ D_H(X_i(n), \partial X_i(n)) \to 0, \\ D_H(X_i, \partial X_i(n)) \to 0, \end{gathered}$$

and thus (v) is false. ∎

3.2. A COLLECTION $\mathcal{J}_1$ RELATED TO A DOMAIN EXCHANGE TRANSFORMATION. Let us define a new collection $\mathcal{J}_1$ that has been studied in many papers ([AI, CS, HZ, Rau, Sie2, SW]).

Let $\gamma_0 = {}^t(1,\ldots,1)$; then $\Gamma_0 := \{z \in \mathbb{Z}^d: (z,\gamma_0) = 0\}$ is a sub-lattice of $\mathbb{Z}^d$ of rank $d-1$. Hence $\Gamma := \pi(\Gamma_0)$ is a full rank lattice on the stable space P. Set

$$\mathcal{J}_1 := \{x + X: x \in \Gamma\} = \Gamma + X.$$

PROPOSITION 3.4: *The collection $\mathcal{J}_1$ is a periodic covering of P.*

Proof: Let us recall that X is defined as the closure of $\pi(Y)$, with $Y = \{f(s_0 s_1 \cdots s_k): k \geq 0\}$, where f denotes the abelianization map and $s_0 \cdots s_n \cdots$ denotes a periodic point of the substitution. We assert that

$$\{x \in \mathbb{Z}^d: \langle x, \gamma_0\rangle \geq 0\} \subseteq \bigcup_{z\in\Gamma_0} (z + Y). \tag{3.10}$$

Suppose $\langle x,\gamma_0\rangle = k \geq 0$. Then $x = (x - f(s_1\cdots s_k)) + f(s_1\cdots s_k)$, where $\langle (x - f(s_1\cdots s_k)), \gamma_0\rangle = 0$ and $f(s_1\cdots s_k) \in Y$.

Since Γ is discrete and $\overline{\pi(Y)} = X$ is bounded, we have

$$P = \pi\Big(\bigcup_{z\in\Gamma_0}(z+Y)\Big) = \bigcup_{z\in\Gamma_0}\overline{\pi(z+Y)} = \Gamma + X.$$

This completes the proof. ∎

PROPOSITION 3.5: *The collection $\mathcal{J}_1$ is a tiling if and only if the collection $\mathcal{J}$ is a tiling.*

Proof: Let P_{γ_0} be the $(d-1)$-dimensional subspace which is orthogonal to γ_0. Let S_{γ_0} be the stepped-surface of P_{γ_0}. Since γ_0 is the diagonal vector, the discrete surface associated with the anti-diagonal plane is made of every half-cube centered in a point of the periodic lattice Γ_0. Hence

$$S_{\gamma_0} = \bigcup_{z\in\Gamma_0}\bigcup_{i=1}^{d}[z, i^*].$$

Let $X^* = \pi(\bigcup_{i=1}^{d}[0, i^*])$. Projecting S_{γ_0} to P, we get $\Gamma + X^*$, which is a lattice tiling of P.

By Proposition 3.4, $\mathcal{J}_1 = \Gamma + X$ is a covering. Hence, it is a tiling if and only if the measure of X is equal to the measure of a fundamental domain of P/Γ, for instance, $X^* = \pi(\bigcup_{i=1}^{d}[0, i^*])$. From Theorem 3.3, $\mu(X) = \mu(X^*)$ if and only if the collection $\mathcal{J}$ is a tiling, which concludes the proof. ∎

Remark 3.6: The lattice Γ is generated by the vectors $\pi(e_i - e_j)$, thus all $\pi(e_i)$ are equivalent mod Γ. As soon as $\mathcal{J}_1$ is a tiling, the dynamical system defined on X by

$$x \mapsto x + \pi(e_i) \pmod{\Gamma}, \quad \text{if } x \in X_i,$$

is obviously isomorphic to a $(d-1)$-dimensional toral translation. This dynamical system is called a domain exchange. Notice that this map is defined since the tiles X_i are disjoint by the tiling hypothesis. However, these tiles appear to be measurably disjoint as soon as the substitution satisfies the so-called strong coincidence combinatorial condition that will be introduced in Section 4.1. Hence, the dynamical system exists even if the collections $\mathcal{J}$ and $\mathcal{J}_1$ are not tilings. If so, it is no longer isomorphic to a toral translation.

3.3. A collection $\mathcal{J}_2$ related to Markov partition.

Recall that π' is the projection from $\mathbb{R}^d$ onto V along P. Let

$$\hat{X}_i := \{X_i - \theta\pi'(e_i) : 0 \le \theta < 1\}, \quad 1 \le i \le d,$$

and let $\hat{X} := \bigcup_{i=1}^{d} \hat{X}_i$. Set

$$\mathcal{J}_2 := \{z + \hat{X}: z \in \mathbb{Z}^d\} = \hat{X} + \mathbb{Z}^d.$$

PROPOSITION 3.7: *The collection $\mathcal{J}_2$ is a tiling of $\mathbb{R}^d$ if and only if $\mathcal{J}$ is a tiling of P.*

Proof: Suppose that $\mathcal{J}$ is a tiling. We consider the intersection of P and $\hat{X}+\mathbb{Z}^d$. By the construction of $\hat{X}$, we see that $\hat{X}_i$ is a tube between two hyperplanes P and $P - e_i$. Pick $x \in \mathbb{Z}^d$; $(x + \hat{X}_i) \cap P$ is not empty if and only if x is between P and $P + e_i$, that is, if and only if $[x, i^*]$ belongs to the stepped-surface S. Moreover, the intersection is $\pi(x) + X_i$ if it is not empty. Therefore, if the collection $\mathcal{J}_2$ is a tiling, then the collection $\mathcal{J}$ is also a tiling.

Conversely, if $\mathcal{J} = \bigcup_{[x,i^*]\in S} \pi(x)+X_i$ is a tiling of P, then by the same reason as above, P is "tiled" by $\hat{X}+\mathbb{Z}^d$ in the sense that $P \subset \hat{X}+\mathbb{Z}^d$ and almost every point of P is covered only once by $\hat{X} + \mathbb{Z}^d$. By translation, we see that for any integer point $z \in \mathbb{Z}^d$, $P + z$ is "tiled" by $\hat{X} + \mathbb{Z}^d$. Since $\pi'(\mathbb{Z})^d$ is dense in V, so $P + \mathbb{Z}^d$ is dense in $\mathbb{R}^d$. It follows that $\hat{X} + \mathbb{Z}^d$ is a tiling of $\mathbb{R}^d$. ∎

Remark 3.8: We may regard the incidence matrix M as a group automorphism on d-dimensional torus. When $(\hat{X}, \mathbb{Z}^d)$ is a tiling, then $\{\hat{X}_1, \ldots, \hat{X}_d\}$ is a Markov partition of the group automorphism M. This result is proved for some given substitutions in [Rau] and [IO], and a class of substitutions in [Pra]. It is announced in [AI] and proved in unpublished [Sie2].

From Proposition 3.5 and Proposition 3.7, we have

THEOREM 1.2: *The three collections $\mathcal{J}$, $\mathcal{J}_1$ and $\mathcal{J}_2$ are tilings simultaneously if one of them is a tiling.*

4. Super-coincidence condition

4.1. COINCIDENCE AND SUPER-COINCIDENCE. Let σ be a substitution over $\mathcal{A} = \{1, \ldots, d\}$. Two distinct letters $i, j \in \mathcal{A}$ are said to have **strong coincidence** if there exist integers k, n such that $\sigma^n(i)$ and $\sigma^n(j)$ have the same k-th letter, and the prefixes of length $k - 1$ of $\sigma^n(i)$ and $\sigma^n(j)$ have the same image under the abelianization map f. A substitution σ is said to satisfy **the strong coincidence condition** if i, j have coincidence for any $i, j \in \mathcal{A}$. The strong coincidence condition for Pisot substitution was introduced first in [Hos2] and then formally defined in [AI].

It is conjectured that every irreducible Pisot substitution satisfies the strong coincidence condition ([BD]). Barge and Diamond showed that the conjecture is true in the two-letter case.

THEOREM 4.1 ([BD]): *Let σ be an irreducible Pisot substitution. Then there exist distinct letters $i, j \in \mathcal{A}$ which have strong coincidence. Consequently, Pisot substitutions over two letters always satisfy the strong coincidence condition.*

Coincidence of two segments (x, i) and (y, j). Following the notations introduced in Section 2, strong coincidences can be expressed in terms of strands. Indeed, two letters have strong coincidences if and only if $F_\sigma^n(0, i) \cap F_\sigma^n(0, j) \neq \emptyset$, that is, if they have a common segment, where $(x, i) \in \mathcal{G}$ denotes a segment starting from x and ending in $x + e_i$ and F_σ denotes the inflation and substitution map associated with σ. Let us extend the definition of coincidence to two segments in $\mathcal{G}$. We say two segments (x, i) and (y, j) in $\mathcal{G}$ have **coincidence** if there exists $n > 0$ such that $F_\sigma^n(x, i) \cap F_\sigma^n(y, j) \neq \emptyset$.

Height. Two segments (x, i) and (y, j) are said to have the same **height** if $(\pi'(x, i))^\circ \cap (\pi'(y, j))^\circ \neq \emptyset$. Clearly $(\pi'(x, i))^\circ \cap (\pi'(y, j))^\circ \neq \emptyset$ if and only if $\{\pi'(F_\sigma^n(x, i))\}^\circ \cap \{\pi'(F_\sigma^n(y, j))\}^\circ \neq \emptyset$. Hence, coincidence between (x, i) and (y, j) implies that they have the same height.

Super-coincidence. Now we are in position to define the super-coincidence condition.

Definition 4.2: A substitution σ is said to satisfy the **super-coincidence condition**, if two segments (x, i) and (y, j) in $\mathcal{G}$ have coincidences whenever they have the same height.

Super-coincidence is the strongest coincidence one can have.

4.2. COINCIDENCE CRITERION FOR TILING. Some criteria for tilings have been established by Siegel [Siel] and Thuswaldner [Thu]. Concerning substitutions associated with a β-numeration system, important conditions for tilings are also given in [Ak2] in terms of the W-property. All these criteria are arithmetics. However, we will give a new criterion for tilings which is purely combinatorial. In this subsection we assume that σ is an irreducible and unimodular Pisot substitution.

LEMMA 4.3: *Let $[x, i^*]$ and $[y, j^*]$ be two elements of the stepped-surface S. If $(-x, i)$ and $(-y, j)$ have coincidences, then $\pi(x) + X_i$ and $\pi(y) + X_j$ do not overlap.*

Proof: If $(-x,i)$ and $(-y,j)$ have coincidences, then there exist n and $(t',k) \in \mathcal{G}$ such that

$$(t',k) \in F_\sigma^n(-x,i) \cap F_\sigma^n(-y,j).$$

From $(t',k) \in F_\sigma^n(-x,i) = -M^n(x) + F_\sigma^n(0,i)$ we infer that $(M^n(x+t),k) \in F_\sigma^n(0,i)$, where $t = M^{-n}t'$. Hence by the definition of dual substitution, we have

$$[x+t,i^*] \in (\sigma^n)^*[0,k^*] = (\sigma^*)^n[0,k^*]. \tag{4.1}$$

Likewise,

$$[y+t,j^*] \in (\sigma^n)^*[0,k^*] = (\sigma^*)^n[0,k^*]. \tag{4.2}$$

Formulas (4.1), (4.2) and Theorem 2.4 imply that $\pi(x+t)+X_i$ and $\pi(y+t)+X_j$ do not overlap. Hence $\pi(x)+X_i$ and $\pi(y)+X_j$ do not overlap. ∎

Particularly, if σ satisfies the strong coincidence condition, then $X = \bigcup_{i=1}^d X_i$ is a non-overlapping union, which is a result of [AI].

PROPOSITION 4.4: *The following statements are equivalent.*

(i) *$\mathcal{J}$ is a tiling.*

(ii) *If $[x,i^*]$ and $[y,j^*]$ belong to the stepped-surface S, then $(-x,i)$ and $(-y,j)$ have coincidences.*

Proof: (ii)⇒ (i) is done by Lemma 4.3. It remains to show that (i)⇒ (ii).

Assume that $\mathcal{J}$ is a tiling. Let $[x,i^*],[y,j^*] \in S$, and let $B(0,r)$ be a ball which contains both x and y. By Theorem 3.3(iii), if we choose n large enough, then the radius of the largest ball B with $S_B \subseteq (\sigma^*)^n[0,1^*]$ can be arbitrarily large. By the quasi-periodicity of the stepped-surface, a translation of $S_{B(0,r)}$ appears in S_B and thus appears in $(\sigma^*)^n[0,1^*]$. So there exists $t \in \mathbb{Z}^d$ such that $[x+t,i^*]$ and $[y+t,j^*]$ belong to $(\sigma^*)^n[0,1^*]$. Reversing the argument of Lemma 4.3, we have

$$(M^n t,1) \in F_\sigma^n(-x,i) \cap F_\sigma^n(-y,j).$$

Hence $(-x,i)$ and $(-y,j)$ have coincidences. ∎

LEMMA 4.5: *Two segments (x,i) and (y,j) have the same height if and only if there exists $t \in \mathbb{Z}^d$ such that $[-x+t,i^*],[-y+t,j^*] \in S$.*

Proof: Notice that $[z,i^*] \in S$ if and only if $(z-e_i,i)$ intersects P, and if and only if $(-z,i)$ intersects P. The first equivalence follows from the definition of stepped-surface, and the second follows the fact that $(z - e_i, i)$ and $(-z, i)$ are symmetric to the origin.

If $[-x+t, i^*], [-y+t, j^*] \in S$, then both $(x-t, i)$ and $(y-t, j)$ intersect P. So (x, i) and (y, j) have the same height.

Conversely, if (x, i) and (y, j) have the same height, then there exists $t \in \mathbb{Z}^d$ such that $(x-t, i)$ and $(y-t, j)$ intersect P. Therefore $[-x+t, i^*]$ and $[-y+t, j^*]$ are elements of S. ∎

THEOREM 1.3: *The collection $\mathcal{J}$ is a tiling if and only if σ satisfies the super-coincidence condition.*

Proof: Suppose that σ satisfies the super-coincidence condition. Then Proposition 4.4 (ii) holds, so that $\mathcal{J}$ is a tiling.

Suppose that $\mathcal{J}$ is a tiling. Take any $(x, i), (y, j)$ which have the same height. Then there is a integer t such that $[-x+t, i^*], [-y+t, j^*] \in S$ (Lemma 4.5), and so that $(x-t, i)$ and $(y-t, j)$ have coincidences (Proposition 4.4). Therefore (x, i) and (y, j) have coincidences. ∎

4.3. PISOT SUBSTITUTIONS OVER TWO LETTERS. A substitution is said to satisfy the **intersection condition** if for any segments (x, i) and (y, j) having the same height, there exists a number n such that $\sigma^n(x, i)$ and $\sigma^n(y, j)$ intersect: precisely, there exist z and k_1, k_2, such that $(z, k_1) \in \sigma^n(x, i), (z, k_2) \in \sigma^n(y, j)$. Obviously, σ satisfies the super-coincidence condition if and only if σ satisfies both the strong coincidence condition and the intersection condition.

As we have seen in [BD], the strong coincidence condition is hard to verify. The intersection condition is also difficult to check. For example, every substitution coming from β-numeration satisfies the strong coincidence condition. In this case the intersection condition is equivalent to the W-property in [Ak2], which is tedious to verify (see [ARS]).

However, it can be shown that Pisot substitutions over two letters satisfy the intersection condition.

THEOREM 1.4: *Pisot substitutions (not necessarily unit) over two letters satisfy the super-coincidence condition.*

Proof: Let σ be a Pisot substitution over two letters; then it must be primitive as we have mentioned in Section 1. By Theorem 4.1, it suffices to show that σ satisfies the intersection condition.

We will use $\overline{F_\sigma^n(x,i)}$ to denote the broken line which is the union of all segments in $F_\sigma^n(x,i)$. By the Pisot property, it is easy to show that

$$\lim_{n\to\infty} \pi\{z\colon z \in \overline{F_\sigma^n(x,i)} \cap \mathbb{Z}^d\} = \lim_{n\to\infty} \pi\{z\colon z \in \overline{F_\sigma^n(0,i)} \cap \mathbb{Z}^d\} = X.$$

Moreover, let L be a broken line segment which is a connected subset of $\overline{F_\sigma^n(x,i)}$; then

$$\lim_{|L|\to\infty} \pi\{z\colon z \in L \cap \mathbb{Z}^d\} = X, \tag{4.3}$$

where $|L|$ denotes the length of L.

Suppose that (x,i) and (y,j) have the same height and do not have coincidence. Then for any $m > 0$, $\overline{F_\sigma^m(x,i)}$ and $\overline{F_\sigma^m(y,j)}$ do not intersect each other. We choose a segment $(z,1) \in F_\sigma^m(x,i)$ satisfying:

(i) $\pi'(z,1) \subset \pi'(\sigma^m(y,j))$;

(ii) there is an element $(z',1) \in \sigma^m(y,j)$ such that z and z' have the same x-coordinate.

Such a segment exists if we choose m large. Without loss of generality we assume that the y-coordinate of z is larger than that of z', and we say $(z,1)$ is "above" $(z',1)$. Now we iterate both $(z,1)$ and $\sigma^m(y,j)$. By the choice of $(z,1)$, we know that $\overline{F_\sigma^N(z,1)}$ does not intersect $\overline{F_\sigma^{m+N}(y,j)}$ and it is always "above" $\overline{F_\sigma^{m+N}(y,j)}$. Take the segments in $\overline{F_\sigma^{m+N}(y,j)}$ which are exactly under $\overline{F_\sigma^N(z,1)}$, and denote the union by L. Then the Hausdorff metric between $\pi\{z\colon z \in \overline{F_\sigma^N(z,1)} \cap \mathbb{Z}^d\}$ and $\pi\{z\colon z \in L \cap \mathbb{Z}^d\}$ is no less than the length of $\pi(e_2)$. On the other hand, both of them should converge to X in Hausdorff metric as $|L|$ tends to infinity. This contradiction proves the theorem. ∎

Combining Theorem 1.3 and Theorem 1.4, we have that $\mathcal{J}$ is a tiling for every unimodular Pisot substitution over two letters. In [BR], we will show that the super-coincidence condition implies pure discrete spectrum. Hence every Pisot substitution over two letters has pure discrete spectrum, which is a result of Hollander and Solomyak [HS].

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